CHAPTER 2

ON ESSENTIAL RIGHT IDEALS AND RIGHT SINGULAR IDEALS

2.1 Introduction

In this chapter, we shall study some properties of essential right ideals and right singular ideals. These ideals are imperative in our study of right finite dimensional and right nonsingular rings. In addition, we shall also study uniform, complement and dense right ideals, and their relations with essential right ideals.

We begin in section two by giving the definitions of the ideals mentioned in the preceding paragraph. In addition, we shall also give the definitions of right finite dimensional and right nonsingular rings.

Next in section three we shall study some characteristics of essential right ideals. We shall also look at essential right ideals in polynomial rings and group rings. Specifically, we show that essentiality is inherited in polynomial rings and group rings (see Propositions 2.3.5 and 2.3.7).

In section four we study some characteristics of right singular ideals and some relations between right singular ideals and essential right ideals. Furthermore, we shall discuss some results on right singular ideals in group rings. It is worth noting here that right singular ideals first appeared in a 1951 paper by Johnson [Jo]. This paper of Johnson set the stage for a deep investigation into the structure of maximal ring of quotients via right singular ideals. Right singular ideals have also been important objects of study because of their close association with zero divisors.

Finally in section five we study some relations between uniform right ideals, complement right ideals, dense right ideals and essential right ideals. Among other things, we shall see that every essential right ideal of a ring R is dense if and only if R is nonsingular (see Propositon 2.5.7).

2.2 Definitions

Let R be a ring and $L \subseteq M$ be R-modules. We say that L is *essential* in M, and write L *ess* M, if L has nonzero intersection with every nonzero submodule of M. If M does not contain an infinite direct sum of nonzero submodules, we say that M is *finite dimensional*. A ring R is said to be *right finite dimensional* if it is finite dimensional when considered as a right R-module. A nonzero R-module U is *uniform* if it does not contain any direct sum of two nonzero R-submodules. Clearly, U is a uniform R-module if and only if the intersection of any two nonzero R-submodules of U is nonzero. Equivalently, we can say that every nonzero R-submodule of U is essential in U.

Let K, I be nonzero right ideals of R. We say that K is a *relative complement* of I if K is maximal with respect to those right ideals having zero intersection with I. Moreover, K is called a *complement right ideal* of R if K happens to be a relative complement of a nonzero right ideal of R. Clearly a complement right ideal of R is not essential in R. Note that the word "right" in all the definitions given above can also be replaced by "left".

If I is a nonzero right ideal of R and $x \in R$, then the *residual* $x^{-1}I$ is given by $x^{-1}I = \{a \in R \mid xa \in I\}$. Thus $x^{-1}I$ is the largest subset K of R such that $xK \subseteq I$. Clearly $x^{-1}I$ is a right ideal of R. If D is a right ideal of R, then D is *dense* if $(x^{-1}D)^{l} = \{0\}$ for all $x \in R$. It is clear from this definition that D is dense if and only if for all $x, y \in R$, $y \neq 0$, there exists an element $a \in R$ such that $xa \in D$ and $ya \neq 0$.

The *right singular ideal* of a ring R is defined to be $Z_r(R) = \{a \in R \mid a^r \ ess \ R\}$. If $Z_r(R) = \{0\}$, then R is a *right nonsingular ring*. Similarly, we define the *left singular ideal* of R as $Z_l(R) = \{a \in R \mid a^l \text{ is a left essential ideal of } R\}$ and R is a left nonsingular ring if $Z_l(R) = \{0\}$. In view of the fact that a subring S of R is itself a ring (not necessarily with identity), we may also have $Z_r(S) = \{a \in S \mid a_S^r \ ess \ S\}$, where $a_S^r = \{x \in S \mid ax = 0\}$; that is, $a_S^r = a^r \cap S$. We shall continue using a^r to denote the right annihilator of an element $a \in R$ except in places where a misunderstanding might occur. In this chapter, since we are only concerned with right singular ideals, we shall use $Z(R)$ to denote $Z_r(R)$.

2.3 Essential right ideals

We begin this section with some basic results on essential right ideals.

Proposition 2.3.1

Let R be a ring and let K, L, L_1, …, L_n be nonzero right ideals of R.

(i) If L *ess* R and $L \subseteq K$, then K *ess* R.

(ii) Let $K \subseteq L$. If K *ess* L and L *ess* R, then K *ess* R.

(iii) If L *ess* R, then $(K \cap L)$ *ess* K.

(iv) Let L_i *ess* R for $i = 1, \ldots, n$. If $L = \bigcap_{i=1}^{n} L_i$, then L *ess* R.

(v) If L *ess* R and $x \in R$, then $x^{-1}L$ *ess* R.

Proof:

(i) Let J be a nonzero right ideal of R. Since L *ess* R, $J \cap L \neq \{0\}$. Then since $L \subseteq K$, so $\{0\} \neq J \cap L \subseteq J \cap K$. Hence, K *ess* R.

(ii) Suppose that $K \cap I = \{0\}$ for some right ideal I of R. Then $K \cap (I \cap L) = (K \cap I) \cap L = \{0\}$. Since $I \cap L \subseteq L$ and K *ess* L, we have $I \cap L = \{0\}$. But L *ess* R implies that $I = \{0\}$. Hence, K *ess* R and we have proved the transitivity of *ess*.

(iii) Let I be a nonzero right ideal of R contained in K. Since L *ess* R, we have $I \cap L \neq \{0\}$. Thus $I \cap (K \cap L) = (I \cap K) \cap L = I \cap L \neq \{0\}$. Since I is arbitrary, so $(K \cap L)$ *ess* K.

(iv) It suffices to prove for the case $n = 2$. Suppose that $(L_1 \cap L_2) \cap J = \{0\}$ for some right ideal J of R. Since L_1 *ess* R and $L_2 \cap J$ is a right ideal of R, so $L_2 \cap J = \{0\}$. Then since L_2 *ess* R, it follows that $J = \{0\}$. Hence, $(L_1 \cap L_2)$ *ess* R.

(v) Clearly $x^{-1}L = \{a \in R \mid xa \in L\}$ is a right ideal of R. Let I be a nonzero right ideal of R. If $xI = \{0\}$, then $I \subseteq x^{-1}L$ and we have $I \cap x^{-1}L \neq \{0\}$. On the other hand, if $xI \neq \{0\}$, then $xI \cap L \neq \{0\}$ since L *ess* R. Thus $I \cap x^{-1}L \neq \{0\}$ by the definition of $x^{-1}L$. Hence, $x^{-1}L$ *ess* R. ■

Proposition 2.3.2

Let $\{R_i\}_{i\geq 1}$ be a family of rings and let L_i be a nonzero right ideal of R_i with L_i *ess* R_i, $i \geq 1$. If $L = \dot{\oplus}_{i\geq 1} L_i$ and $R = \dot{\oplus}_{i\geq 1} R_i$, then L *ess* R.

Proof: Let J be a nonzero right ideal of R and let $a = (x_i)_{i\geq 1}$ be a nonzero element of J, where $x_i \in R_i$ for $i \geq 1$. Since $x_i = 0$ for all but a finite number of i, there exists a positive integer n such that $x_i = 0$ for $i > n$. If $x_i \in L_i$ for all i, then the proof is complete. Otherwise, we choose a smallest k_1, $k_1 \geq 1$, such that $x_{k_1} \notin L_{k_1}$. Since $x_{k_1} \neq 0$ and L_{k_1} *ess* R_{k_1}, it follows that $x_{k_1}R_{k_1} \cap L_{k_1} \neq \{0\}$. So $0 \neq x_{k_1}r_1 \in L_{k_1}$ for some $r_1 \in R_{k_1}$.

Next we consider $ar_1 = (x_ir_1)_{i\geq 1}$. Note that $x_ir_1 \in L_i$ for $i = 1, \ldots, k_1$ and $x_jr_1 \in R_j$ for $j \geq k_1 + 1$. Again, we choose the smallest k_2, $k_2 > k_1$, such that $x_{k_2}r_1 \notin L_{k_2}$. By the same argument, we have $0 \neq x_{k_2}r_1r_2 \in L_{k_2}$ for some $r_2 \in R_{k_2}$. Now consider $ar_1r_2 = (x_ir_1r_2)_{i\geq 1}$. Note that $x_ir_1r_2 \in L_i$ for $i = 1, \ldots, k_2$ and $x_jr_1r_2 \in R_j$ for $j \geq k_2 + 1$. By choosing the smallest k_3, $k_3 > k_2$, such that $x_{k_3}r_1r_2 \notin L_{k_3}$ and repeating the same process as above, we finally obtain for some l $(1 \leq l \leq n)$ a nonzero element $ar_1 \cdots r_l \in J \cap L$. Since J is arbitrary, it follows that L *ess* R. ■

Throughout the rest of this section, L will denote a nonzero right ideal of a ring R.

Proposition 2.3.3

L *ess* R if and only if for any nonzero elements $a_1, \ldots, a_n \in R$, there exists an element $b \in R$ such that $a_i b \in L$ for all $i = 1, \ldots, n$ with some of the $a_i b$'s not equal to zero.

Proof: Assume that L *ess* R. Let $a_1, \ldots, a_n$ be nonzero elements of R. If $a_i \in L$ for all $i = 1, \ldots, n$, we take $b = 1 \in R$ and the proof is complete. Otherwise, we choose the smallest k_1, $1 \le k_1 \le n$, such that $a_{k_1} \notin L$. Since $a_{k_1} \ne 0$ and L *ess* R, we have $a_{k_1} R \cap L \ne \{0\}$; thus there exists an element $b_1 \in R$ such that $0 \ne a_{k_1} b_1 \in L$.

Next, we consider $a_1 b_1, \ldots, a_{k_1} b_1, \ldots, a_n b_1 \in R$. If $a_i b_1 \in L$ for all $i = 1, \ldots, n$, we take $b = b_1$ and the proof is complete. Otherwise, we choose the smallest k_2, $k_1 < k_2 \le n$ such that $a_{k_2} b_1 \notin L$. By the same argument as in the preceding paragraph, there is some $b_2 \in R$ with $0 \ne a_{k_2} b_1 b_2 \in L$. Now consider $a_1 b_1 b_2, \ldots, a_{k_2} b_1 b_2, \ldots, a_n b_1 b_2 \in R$. By proceeding in the same way as above, we eventually obtain an element $b = b_1 \cdots b_l \in R$, $1 \le l \le n$, such that $a_i b \in L$ for all $i = 1, \ldots, n$, with some of the $a_i b$'s not equal to zero.

Conversely, let I be a nonzero right ideal of R. Then for any nonzero element $a \in I \subseteq R$, there exists an element $b \in R$ such that $0 \ne ab \in L$. Since ab is also contained in I, so $L \cap I \ne \{0\}$. Then since I is arbitrary, it follows that L *ess* R. ■

A useful consequence of the above proposition is as follows:

Corollary 2.3.4

L *ess* R if and only if for any nonzero element a of R, there exists an element $b \in R$ such that ab is nonzero and $ab \in L$.

We now move on to some results on essential right ideals in polynomial rings and group rings.

Proposition 2.3.5

L *ess* R if and only if $L[x_1, x_2, \ldots]$ *ess* $R[x_1, x_2, \ldots]$.

Proof: Assume that L *ess* R. Let p be a nonzero element of $R[x_1, x_2, \ldots]$ and let $a_1, \ldots, a_n \in R$ be the nonzero coefficients of p. Since L *ess* R, it follows from Proposition 2.3.3 that there exists an element $b \in R$ such that $a_i b \in L$ for all $i = 1, \ldots, n$ with some of the $a_i b \neq 0$. Consequently, $pb \neq 0$ and $pb \in L[x_1, x_2, \ldots]$. Thus $L[x_1, x_2, \ldots]$ *ess* $R[x_1, x_2, \ldots]$.

Conversely, let a be a nonzero element of R. Again, from Proposition 2.3.3, $L[x_1, x_2, \ldots]$ *ess* $R[x_1, x_2, \ldots]$ and $a \in R \subseteq R[x_1, x_2, \ldots]$ imply that there exists an element $q \in R[x_1, x_2, \ldots]$ such that $0 \neq aq \in L[x_1, x_2, \ldots]$. It follows that there is a nonzero coefficient b of q such that $ab \neq 0$ and $ab \in L$. Therefore L *ess* R, as required. ∎

Corollary 2.3.6

L *ess* R if and only if $L[x]$ *ess* $R[x]$.

Proposition 2.3.7

Let R be a ring and let G be a group. Then L *ess* R if and only if LG *ess* RG.

Proof: Assume that L *ess* R. Let r be a nonzero element of RG, say $r = r_{g_1}g_1 + \cdots + r_{g_n}g_n$, where $r_{g_i} \in R$ and $g_i \in G$ for all $i = 1, \ldots, n$. Since $L\ ess\ R$, it follows from Proposition 2.3.3 that there exists an element $b \in R$ such that $r_{g_i}b \in L$ for all $i = 1, \ldots, n$ with $r_{g_j}b \neq 0$ for some j. It follows that $0 \neq rb \in LG$ and hence, $LG\ ess\ RG$.

Conversely, let a be a nonzero element of R. Since LG *ess* RG and $a \in R \subseteq RG$, there is an element $s \in RG$ such that $0 \neq as \in LG$. If $s = s_{g_1}g_1 + \cdots + s_{g_n}g_n$, then $as_{g_i} \in L$ for all $i = 1, \ldots, n$ with $as_{g_j} \neq 0$ for some j. Hence, L *ess* R. ■

The necessity part in Propositon 2.3.7 is in fact a special case of the following result:

Proposition 2.3.8

Let R be a ring and G a group. Let H be a normal subgroup of G and L' a nonzero right ideal of RH. If L' *ess* RH, then $L'RG$ *ess* RG.

Proof: Let r be a nonzero element of RG, say $r = r_1g_1 + \cdots + r_ng_n$, where $g_i \in G$, $r_i \in R$ and $r_i \neq 0$ for all $i = 1, \ldots, n$. We wish to show that there exists an element $b \in RG$ such that $0 \neq rb \in L'RG$. If $r_ig_i \in L'RG$ for all i, then the proof is complete. Otherwise, we choose a smallest k_1, $1 \leq k_1 \leq n$, such that $r_{k_1}g_{k_1} \notin L'RG$. Since $L'\ ess\ RH$ and $r_{k_1} \in R \subseteq RH$, it follows from Proposition 2.3.3 that there exists an element $b_1 = \sum_{i=1}^{m_1} b_{1_i} h_i \in RH$ so that $0 \neq r_{k_1} b_1 = \sum_{i=1}^{m_1} (r_{k_1} b_{1_i}) h_i \in L'$. Note that H being normal implies that there exists $h_i' \in H$ so that $h_i g_{k_1} = g_{k_1} h_i'$, $i = 1, \ldots, m_1$. Let $b_1' = \sum_{i=1}^{m_1} b_{1_i} h_i' \in RH$. Then $g_{k_1} b_1' = b_1 g_{k_1}$ and hence, $r_{k_1} g_{k_1} b_1' = r_{k_1} b_1 g_{k_1} \in L'RG$. Now consider

$$rb_1' = r_1 g_1 b_1' + \cdots + r_{k_1} g_{k_1} b_1' + \cdots + r_n g_n b_1'.$$

Write $rb_1' = r_1 d_1' + \cdots + r_{k_1} d_{k_1}' + \cdots + r_n d_n'$, where $d_i' = g_i b_1'$ for $i = 1, \ldots, n$. Note that $r_i d_i' \in L'RG$ for $i = 1, \ldots, k_1$ and $r_j d_j' \in RG$ for $j = k_1 + 1, \ldots, n$. Again, we choose the smallest k_2, $k_1 + 1 \leq k_2 \leq n$ such that $r_{k_2} d_{k_2}' \notin L'RG$. By the same argument as above, there exist $b_2, b_2' \in RH$ so that $0 \neq r_{k_2} b_2 \in L'$ and $d_{k_2}' b_2' = b_2 d_{k_2}'$. It follows that $r_{k_2} d_{k_2}' b_2' = r_{k_2} b_2 d_{k_2}' \in L'RG$. Now consider $rb_1' b_2' = r_1 d_1' b_2' + \cdots + r_{k_2} d_{k_2}' b_2' + \cdots + r_n d_n' b_2'$. Write $rb_1' b_2' = r_1 d_1'' + \cdots + r_{k_2} d_{k_2}'' + \cdots + r_n d_n''$, where $d_i'' = d_i' b_2'$ for $i = 1, \ldots, n$. Note that $r_i d_i'' \in L'RG$ for $i = 1, \ldots, k_2$ and $r_j d_j'' \in RG$ for $j = k_2 + 1, \ldots, n$. By using the same argument as above and repeating this process, we eventually have an element $b = b_1' \cdots b_l' \in RH \subseteq RG$ $(1 \leq l \leq n)$ such that $0 \neq rb \in L'RG$. Hence $L'RG \text{ ess } RG$, as required. ■

We note that Proposition 2.3.8 has been proven in the more general setting of modules by Burgess [Bu, p.867].

Proposition 2.3.9

Let $\{R_i\}_{i\geq 1}$ be a family of rings and let G be a group. Let L_i be a nonzero right ideal of R_i with L_i *ess* R_i, $i \geq 1$. If $R = \dot{\oplus}_{i\geq 1} R_i$, then $\dot{\oplus}_{i\geq 1} L_i G$ *ess* RG.

Proof: Note that $\dot{\oplus}_{i\geq 1} L_i$ *ess* R by Proposition 2.3.2. The result then follows easily from Propositions 2.3.7 and 1.4.3. ■

2.4 Right singular ideals

All singular ideals considered in this section are right singular ideals. In general, the left and right singular ideals of a ring need not be equal. We shall provide an example of this later on in Chapter 4. We begin this section with

Proposition 2.4.1

Let R be a ring and let I be a nonzero right ideal of R.

(i) $Z(R)$ is a two-sided ideal of R.

(ii) $Z(R) \cap I \subseteq Z(I)$. Furthermore, if I *ess* R, then $Z(R) \cap I = Z(I)$.

(iii) Let $R_1, \ldots, R_n$ be a family of rings. If $R = \prod_{i=1}^{n} R_i$, then $Z(R) = \prod_{i=1}^{n} Z(R_i)$.

Proof:

(i) Let $a, b \in Z(R)$. Then a^r *ess* R and b^r *ess* R imply that $(a^r \cap b^r)$ *ess* R by Proposition 2.3.1(iv). Since $a^r \cap b^r \subseteq (a+b)^r$, it follows from Proposition 2.3.1(i) that $(a+b)^r$ *ess* R; hence, $a+b \in Z(R)$. Now let $s \in R$. Note that a^r *ess* R implies that

$s^{-1}a^r$ *ess* R by Proposition 2.3.1(v). Also $as(s^{-1}a^r) \subseteq aa^r = \{0\}$ implies that $s^{-1}a^r \subseteq (as)^r$. It follows that $(as)^r$ *ess* R and so we have $as \in Z(R)$. Since $a^r \subseteq (sa)^r$ implies that $(sa)^r$ *ess* R, we also have that $sa \in Z(R)$. We have thus shown that $Z(R)$ is a two-sided ideal of R.

(ii) Let $a \in Z(R) \cap I$. Then $a \in I$ and a^r *ess* R. From Proposition 2.3.1(iii), we know that $(a^r \cap I)$ *ess* I. Since $a^r \cap I = a_I^r$, we have a_I^r *ess* I. Thus $a \in Z(I)$ and so $Z(R) \cap I \subseteq Z(I)$.

Now assume that I *ess* R. To prove the reverse inclusion, let $a \in Z(I)$. Then a_I^r *ess* I; that is, $(a^r \cap I)$ *ess* I. It follows that $(a^r \cap I)$ *ess* R by Proposition 2.3.1(ii). Since $a^r \cap I \subseteq a^r$, we have a^r *ess* R by Proposition 2.3.1(i). Thus $a \in Z(R) \cap I$ and so $Z(R) \cap I = Z(I)$, as required.

(iii) It suffices to consider the case $n = 2$, say $R = R_1 \times R_2$. Let $x = (x_1, x_2) \in Z(R_1) \times Z(R_2)$. Then x_i^r *ess* R_i, $i = 1, 2$. Let $a = (a_1, a_2) \in R$, where $(a_1, a_2) \neq (0,0)$. Without loss of generality, we may assume that $a_1 \neq 0$. Since x_1^r *ess* R_1, it follows from Corollary 2.3.4 that there exists $b_1 \in R_1$ such that $0 \neq a_1 b_1 \in x_1^r$. By choosing $b = (b_1, 0) \in R$, we have $ab = (a_1 b_1, 0) \in x^r$ and $ab \neq (0,0)$. Thus x^r *ess* R and so $x \in Z(R)$. Hence $Z(R_1) \times Z(R_2) \subseteq Z(R)$.

On the other hand, let $x = (x_1, x_2) \in Z(R)$, where $x_i \in R_i$, $i = 1, 2$. We wish to show that x_i^r *ess* R_i, $i = 1, 2$. Let $0 \neq a_1 \in R_1$. Then $a = (a_1, 0) \in R$ and $a \neq (0,0)$. Since x^r *ess* R, there exists $b = (b_1, b_2) \in R$ such that $ab = (a_1 b_1, 0) \in x^r$ and $ab \neq (0,0)$. That is, $a_1 b_1 \in x_1^r$ and $a_1 b_1 \neq 0$. By Corollary 2.3.4, we know that x_1^r *ess* R_1. Similarly, it can be shown that x_2^r ess R_2. Thus x_i^r ess R_i , which implies that $x_i \in Z(R_i)$, $i = 1, 2$. Hence, $x = (x_1, x_2) \in Z(R_1) \times Z(R_2)$ and it follows that $Z(R) \subseteq Z(R_1) \times Z(R_2)$, as required. ■

By Proposition 2.3.1(iv), we know that the intersection of a finite number of essential ideals is essential. Since

$$Z(R)^r = \bigcap_{a \in Z(R)} a^r$$

and each a^r is essential in R, it is clear that $Z(R)^r$ is essential in R if $Z(R)$ has a finite number of elements. A natural question to ask is under what other conditions is $Z(R)^r$ essential. An answer to this question is provided in the following proposition:

Proposition 2.4.2

If R is a ring satisfying the d.c.c. on right annihilator ideals, then $Z(R)^r$ ess R. Moreover, $Z(R) = Z(R)^{rl}$.

Proof: Let $x_1 \in Z(R)$. It is clear that $x_1^r \supseteq Z(R)^r$. If $x_1^r \neq Z(R)^r$, then there exists $a \in x_1^r$ such that $x_1 a = 0$ but $x_2 a \neq 0$ for some $x_2 \in Z(R)$. Therefore $x_1^r \supsetneqq \{x_1, x_2\}^r$. Again, if $\{x_1, x_2\}^r \neq Z(R)^r$, there exists $x_3 \in Z(R)$ such that $\{x_1, x_2\}^r \supsetneqq \{x_1, x_2, x_3\}^r$. By continuing in this manner, we have a descending chain

$$x_1^r \supsetneqq \{x_1, x_2\}^r \supsetneqq \{x_1, x_2, x_3\}^r \supsetneqq \cdots$$

of right annihilator ideals of R. Since R satisfies the d.c.c. on right annihilator ideals, there must exist a positive integer n such that $\{x_1, \ldots, x_n\}^r = Z(R)^r$. We thus have that

$Z(R)^r = \bigcap_{i=1}^{n} x_i^r$ is an essential right ideal of R. Since $Z(R)^r \subseteq Z(R)^{rlr}$, it follows that $Z(R)^{rlr}$ is also an essential right ideal of R. Hence, $Z(R)^{rl} \subseteq Z(R)$. The reverse inclusion $Z(R) \subseteq Z(R)^{rl}$ is clear. The equality $Z(R) = Z(R)^{rl}$ thus follows. ■

Remark: The left analogue of Proposition 2.4.2 has appeared in [Fi2, Proposition 2.1]. An interesting fact to note from Proposition 2.4.2 is that if R satisfies the d.c.c. on right annihilator ideals, then $Z(R)$ is the left annihilator of an essential right ideal.

Corollary 2.4.3

If R is a semiprime ring satisfying the d.c.c. on right annihilator ideals, then $Z(R) = \{0\}$.

Proof: Suppose that $Z(R) \neq \{0\}$. Since $Z(R)^r \; ess \; R$, so $Z(R) \cap Z(R)^r$ is a nonzero nilpotent ideal of R. This contradicts the fact that R is semiprime. Hence, we must have $Z(R) = \{0\}$. ■

Remark: Fisher [Fi2] has shown that under the hypothesis of Collorary 2.4.3, even the left singular ideal of R is zero.

Proposition 2.4.4

Let L be a two-sided ideal of a ring R. If $Z\left(R/L\right) = R/L$, then $x^{-1}L \; ess \; R$ for any $x \in R$. Furthermore, if $Z(R) = \{0\}$, then $L \; ess \; R$.

Proof: First, we show that if $Z\left(R/L\right) = R/L$ and $x \in R$, then $x^{-1}L \; ess \; R$. Note that for any $\bar{x} = x + L \in R/L$, we have $\bar{x}^r = \{a + L \mid xa \in L, \, a \in R\}$. Since $Z\left(R/L\right) = R/L$, we have $\bar{x}^r \; ess \; R/L$ for any $\bar{x} \in R/L$. By Corollary 2.3.4, this means that for any $\bar{a} = a + L \in R/L$, $a \notin L$, there exists an element $\bar{b} = b + L \in R/L$ such that $\bar{a}\bar{b} = ab + L \neq L$ and $\bar{x}\bar{a}\bar{b} = xab + L = L$; that is, $ab \notin L$ and $xab \in L$. It follows that for any $0 \neq a \in R, \, a \notin L$, there exists an element $b \in R$ such that $0 \neq ab \in x^{-1}L$. Also, if $a \in L$, then there certainly exists an element $b \in R$ such that $ab \in x^{-1}L$. Hence, $x^{-1}L \; ess \; R$.

Now assume that $Z(R) = \{0\}$. Suppose that $L \cap I = \{0\}$ for some right ideal I of R. Let $y \in I$. Then $y^{-1}L \; ess \; R$ as proven earlier. Note that $y(y^{-1}L) \subseteq I$ and $y(y^{-1}L) \subseteq L$ imply that $y(y^{-1}L) \subseteq L \cap I = \{0\}$. Thus $y^{-1}L \subseteq y^r$, which implies that $y^r \; ess \; R$; that is, $y \in Z(R) = \{0\}$. Hence, $I = \{0\}$ and it follows that $L \; ess \; R$. ■

Lemma 2.4.5

Let R be a ring and let G be a group. If $a = \sum_{i=1}^{n} a_{g_i} g_i \in RG$, where $a_{g_i} \in R$, $g_i \in G$, then $\left(\bigcap_{i=1}^{n} a_{g_i}^r\right) G \subseteq a^r$.

Proof: Let $b = \sum_{j=1}^{m} b_{h_j} h_j \in \left(\bigcap_{i=1}^{n} a_{g_i}^r\right) G$, where $b_{h_j} \in \bigcap_{i=1}^{n} a_{g_i}^r$, $h_j \in G$. Then

$$ab = \left(\sum_{i=1}^{n} a_{g_i} g_i\right)\left(\sum_{j=1}^{m} b_{h_j} h_j\right)$$
$$= \sum_{g}\left(\sum_{g_i h_j = g} a_{g_i} b_{h_j}\right) g .$$

Since $b_{h_j} \in \bigcap_{i=1}^{n} a_{g_i}^r$ for $j = 1, \ldots, m$, so $a_{g_i} b_{h_j} = 0$ for all $i = 1, \ldots, n$; $j = 1, \ldots, m$. Thus $ab = 0$, which implies that $b \in a^r$. Hence $\left(\bigcap_{i=1}^{n} a_{g_i}^r\right) G \subseteq a^r$, as desired. ■

Proposition 2.4.6

Let R be a ring and let G be a group. Then $Z(R)G \subseteq Z(RG)$.

Proof: Let $a = a_{g_1} g_1 + \cdots + a_{g_n} g_n \in Z(R)G$, where $a_{g_i} \in Z(R)$ and $g_i \in G$ for all $i = 1, \ldots, n$. Note that $a_{g_i}^r$ *ess* R for all i implies that $\bigcap_{i=1}^{n} a_{g_i}^r$ *ess* R by Proposition 2.3.1(iv). Then by Proposition 2.3.7, $\left(\bigcap_{i=1}^{n} a_{g_i}^r\right) G$ *ess* RG. From the previous lemma, we have that $\left(\bigcap_{i=1}^{n} a_{g_i}^r\right) G \subseteq a^r$. It then follows from Proposition 2.3.1(i) that a^r *ess* RG. Thus $a \in Z(RG)$ and the result follows. ■

In the next proposition we will show that the reverse containment $Z(RG) \subseteq Z(R)G$ also holds if G is a free abelian group. In order to prove this, we first state the following definitions:

Let R be a ring and let G be a free abelian group with free generators x_i, $i \geq 1$. Let $a = \sum_{g_i \in G} a_{g_i} g_i \in RG$, where $a_{g_i} \in R$, $g_i = x_1^{c_{i_1}} \cdots x_{n_i}^{c_{i_{n_i}}}$ for some $c_{i_j} \in \mathbb{Z}$, $j = 1, \ldots, n_i$, $n_i \in \mathbb{N}$. We say that $a_{g_l} g_l$, where $g_l = x_1^{c_{l_1}} \cdots x_{n_l}^{c_{l_{n_l}}}$, is the *leading term* of a if the power of g_l, $c_{l_1} + \cdots + c_{l_{n_l}}$, is maximal among those power of the g_i. That is, if there is some $a_{g_k} g_k$, where $g_k = x_1^{c_{k_1}} \cdots x_{n_k}^{c_{k_{n_k}}}$ such that $c_{k_1} + \cdots + c_{k_{n_k}} = c_{l_1} + \cdots + c_{l_{n_l}}$, then there exists t, $1 \leq t \leq \min(n_k, n_l)$, such that $c_{l_t} > c_{k_t}$ and $c_{l_s} = c_{k_s}$ for all $s < t$. In this case, a_{g_l} is called the *leading coefficient* of a.

Lemma 2.4.7

Let R be a ring and let G be a free abelian group. Let $a = \sum_{g_i \in G} a_{g_i} g_i \in RG$. If a_0 is the leading coefficient of a and $a_0^r \cap I = \{0\}$ for some nonzero right ideal I of R, then $a^r \cap IG = \{0\}$.

Proof: Suppose that $a^r \cap IG \neq \{0\}$. Then there exists $0 \neq b \in IG$ such that $ab = 0$. Let $b_0 (\neq 0)$ be the leading coefficient of b. Then $a_0 b_0$ becomes the leading coefficient of ab. But $ab = 0$ implies that $a_0 b_0 = 0$, that is, $0 \neq b_0 \in a_0^r \cap I$; a contradiction. Therefore, $a^r \cap IG = \{0\}$. ■

Proposition 2.4.8 (Wilkerson, [Wi])

Let R be a ring and let G be a free abelian group. Then $Z(R)G = Z(RG)$.

<u>Proof</u>: From Proposition 2.4.6, we know that $Z(R)G \subseteq Z(RG)$. For the reverse inclusion, let $s = \sum_g s_g g$. Write $s = a + b$, where $a = \sum_{s_g \in Z(R)} s_g g$ and $b = \sum_{s_g \notin Z(R)} s_g g$. Since $a \in Z(R)G \subseteq Z(RG)$ and $s \in Z(RG)$, it follows that $b \in Z(RG)$. We wish to show that $b = 0$. Suppose that $b \neq 0$ and let b_0 be the leading coefficient of b. Since $b_0 \notin Z(R)$, so $b_0^r \cap I = \{0\}$ for some nonzero right ideal I of R. But by Lemma 2.4.7, this implies that $b^r \cap IG = \{0\}$. This contradicts the fact that $b \in Z(RG)$. Therefore, $b = 0$ and we have $s = a \in Z(R)G$. Hence $Z(R)G = Z(RG)$, as required. ■

Corollary 2.4.9

Let R be a ring and let G be an infinite cyclic group. Then $Z(R)G = Z(RG)$.

Proposition 2.4.10

Let $R_1, \ldots, R_n$ be a family of rings and let G be a group. If $R = \prod_{i=1}^{n} R_i$, then $Z(RG) \cong \prod_{i=1}^{n} Z(R_i G)$. In particular, if G is a free abelian group, then $Z(R)G \cong \prod_{i=1}^{n} Z(R_i)G$.

<u>Proof</u>: Note that $RG = \left(\prod_{i=1}^{n} R_i\right) G \cong \prod_{i=1}^{n} R_i G$ by Proposition 1.4.1. The first assertion then follows from Proposition 2.4.1(iii). If G is a free abelian group, then by Proposition 2.4.8, we have that $Z(R)G = Z(RG) \cong \prod_{i=1}^{n} Z(R_i G) = \prod_{i=1}^{n} Z(R_i)G$. ■

2.5 Some relations and applications

We close this chapter with a discussion of conditions under which certain ideals in a ring are essential and some related results. First, we look at

Proposition 2.5.1

Let R be a ring and let U be a uniform right ideal of R. For any nonzero right ideals I, J of R, if $I \cap U \neq \{0\}$ and $J \cap U \neq \{0\}$, then $I \cap J \neq \{0\}$.

Proof: Suppose that $I \cap J = \{0\}$. Then $(I \cap U) \cap (J \cap U) = \{0\}$. Hence, $I \cap U$ and $J \cap U$ are two nonzero right ideals of R with $(I \cap U) \oplus (J \cap U) \subseteq U$. This is a contradiction since U is uniform. Therefore, $I \cap J \neq \{0\}$. ■

Proposition 2.5.2

Let R be a ring and let $U_1, \ldots, U_n$ be uniform right ideals of R so that $\oplus_{i=1}^{n} U_i \; ess \; R$. For any nonzero right ideal I of R, if $I \cap U_i \neq \{0\}$ for all $i = 1, \ldots, n$, then $\oplus_{i=1}^{n} (I \cap U_i) \; ess \; R$. In particular, $I \; ess \; R$.

Proof: Let $A_i = I \cap U_i$ for $i = 1, \ldots, n$. We wish to show that $\oplus_{i=1}^{n} A_i \; ess \; R$. Let J be a nonzero right ideal of R. Since $\oplus_{i=1}^{n} U_i \; ess \; R$, there exists a nonzero element $x \in J \cap \oplus_{i=1}^{n} U_i$. Write $x = u_1 + \cdots + u_n$, where $u_i \in U_i$ for $i = 1, \ldots, n$. If $u_i \in A_i$ for all i, then the proof is complete. Otherwise, we choose the smallest k_1, $1 \leq k_1 \leq n$, such that

$u_{k_1} \notin A_{k_1}$. Since $u_{k_1}R \subseteq U_{k_1}$, $A_{k_1} \subseteq U_{k_1}$ and ${}_{k_1}$ is uniform, we have $u_{k_1}R \cap A_{k_1} \neq \{0\}$. That is, there exists $r_1 \in R$ such that $0 \neq u_{k_1}r_1 \in A_{k_1}$.

Next we note that $xr_1 = u_1r_1 + \cdots + u_{k_1}r_1 + \cdots + u_nr_1$, where $u_ir_1 \in A_i$ for $i = 1, \ldots, k_1$ and $u_jr_1 \in U_j$ for $j = k_1 + 1, \ldots, n$. If $u_jr_1 \in A_j$ for $j = k_1 + 1, \ldots, n$, then we are done. Otherwise, we choose the smallest k_2, $k_1 < k_2 \leq n$, such that $u_{k_2}r_1 \notin A_{k_2}$. By the same argument as above, there exists $r_2 \in R$ such that $0 \neq u_{k_2}r_1r_2 \in A_{k_2}$.

We next consider $xr_1r_2 = u_1r_1r_2 + \cdots + u_{k_2}r_1r_2 + \cdots + u_nr_1r_2$ and repeat the same argument as above. By continuing the process, we will eventually obtain a nonzero element y, where $y = xr_1 \cdots r_l$ for some l ($\leq l \leq n$) and $y \in J \cap \oplus_{i=1}^{n} A_i$. Since J is arbitrary, so $\oplus_{i=1}^{n} A_i$ *ess* R. Moreover, since $\oplus_{i=1}^{n} A_i \subseteq I$, we have by Proposition 2.3.1(i) that I *ess* R. ■

Proposition 2.5.3

Let R be a ring and let K, I be nonzero right ideals of R. If K is a relative complement of I, then $(I \oplus K)$ *ess* R.

Proof: Suppose that $(I \oplus K) \cap J = \{0\}$ for some right ideal J of R. Then $I \cap J = \{0\}$ and $K \cap J = \{0\}$. We wish to show that $I \cap (K \oplus J) = \{0\}$. Suppose that $I \cap (K \oplus J) \neq \{0\}$. Then there is some nonzero element $i \in I$ such that $i = k + j \in K \oplus J$ for some nonzero element $k \in K$ and some nonzero element $j \in J$. It follows that $j = i - k \in (I \oplus K) \cap J$, which implies that $(I \oplus K) \cap J \neq \{0\}$; a contradiction. Thus we must have

$I \cap (K \oplus J) = \{0\}$. Since K is a relative complement of I, so $K \oplus J = K$ which implies that $J = \{0\}$. Therefore, $(I \oplus K)$ *ess* R. ■

As an immediate consequence, we have

Proposition 2.5.4

Let R be a ring and let G be a group. Let K, I be nonzero right ideals of R. If K is a relative complement of I, then $(IG \oplus KG)$ *ess* RG.

Proof: This follows from Propositions 1.4.2, 2.3.7 and 2.5.3. ■

Before proceeding, we look at some properties of dense right ideals. These properties can also be found in Passman [Pa3].

Proposition 2.5.5

Let R be a ring, let D, D' be dense right ideals of R and I a right ideal of R.

(i) If $D \subseteq I$, then I is dense.

(ii) If $x \in R$, then $x^{-1}D$ is dense.

(iii) $D \cap D'$ is dense.

(iv) D *ess* R.

(v) If I is a two-sided ideal of R, then I is dense if and only if $I^{l} = \{0\}$.

Proof:

(i) Since $D \subseteq I$ implies that $x^{-1}D \subseteq x^{-1}I$ for all $x \in R$, it follows that $(x^{-1}I)^{l} \subseteq (x^{-1}D)^{l} = \{0\}$ for all $x \in R$. Hence $(x^{-1}I)^{l} = \{0\}$ for all $x \in R$, as required.

(ii) Let $y \in R$. First, we note that $a \in y^{-1}(x^{-1}D)$ if and only if $ya \in x^{-1}D$ and hence, if and only if $xya \in D$. Thus $y^{-1}(x^{-1}D) = (xy)^{-1}D$. We then have that $\left(y^{-1}(x^{-1}D)\right)^{l} = \left((xy)^{-1}D\right)^{l} = \{0\}$ since D is dense. Since $y \in R$ is arbitrary, it follows that $x^{-1}D$ is dense.

(iii) Fix $x \in R$ and let $a \in \left(x^{-1}(D \cap D')\right)^{l}$. We first note that $x^{-1}(D \cap D') = x^{-1}D \cap x^{-1}D'$. This is clear since $b \in x^{-1}(D \cap D')$ if and only if $xb \in D \cap D'$ if and only if $b \in x^{-1}D \cap x^{-1}D'$. Now, for any $y \in x^{-1}D'$, let $J = (xy)^{-1}D = y^{-1}(x^{-1}D)$. Then $yJ \subseteq x^{-1}D$ and $yJ \subseteq x^{-1}D'$ since $y \in x^{-1}D'$. Thus $yJ \subseteq x^{-1}D \cap x^{-1}D' = x^{-1}(D \cap D')$ and we have $ayJ = \{0\}$. But D being dense implies that $J^{l} = \{0\}$; hence, $ay = 0$. Since this holds for all $y \in x^{-1}D'$, it follows that $a \in (x^{-1}D')^{l} = \{0\}$. Therefore, $D \cap D'$ is dense.

(iv) Suppose that $D \cap J = \{0\}$ for some right ideal J of R. If $x \in J$, then $x(x^{-1}D) \subseteq D \cap J = \{0\}$. Thus $x \in (x^{-1}D)^{l} = \{0\}$. Since $x \in J$ is arbitrary, so $J = \{0\}$ and hence, D *ess* R as required.

(v) Note that I being a two-sided ideal of R implies that $I \subseteq x^{-1}I$ for all $x \in R$. Thus $(x^{-1}I)^{l} \subseteq I^{l}$ for all $x \in R$. It follows that if $I^{l} = \{0\}$, then $(x^{-1}I)^{l} = \{0\}$ for all $x \in R$;

hence I is dense in R. If I is dense in R, then $(x^{-1}I)^l = \{0\}$ for all $x \in R$. In particular, $I^l = \{0\}$. ■

Proposition 2.5.6

Let R be a ring and let G be a group. Then D is a dense right ideal of R if and only if DG is a dense right ideal of RG.

<u>Proof</u>: First, we note by Proposition 2.5.5(iv) that every dense right ideal of R is essential in R. Let $x' = \sum_{i=1}^{m} x_i g_i$, $y' = \sum_{j=1}^{n} y_j g_j \in RG$, where x_i, $y_j \in R$, g_i, $g_j \in G$ and where not all $y_j = 0$. Assume without loss of generality that $y_1 \neq 0$. Since D is dense, it follows by induction that there exists an element $a \in R$ such that $y_1 a \neq 0$ and $x_i a \in D$ for all $i = 1, \ldots, m$. Consequently, $x'a = \sum_{i=1}^{m}(x_i a)g_i \in DG$ and $y'a = \sum_{j=1}^{n}(y_j a)g_j \neq 0$. Hence, DG is a dense right ideal of RG.

Conversely, let $x, y \in R, y \neq 0$. Since DG is dense, there exists $a' = \sum_{i=1}^{m} a_i g_i \in RG$ so that $xa' = \sum_{i=1}^{m}(xa_i)g_i \in DG$ and $ya' = \sum_{i=1}^{m}(ya_i)g_i \neq 0$. That is, $xa_i \in D$ for all $i = 1, \ldots, m$ and $ya_k \neq 0$ for some k, $\leq k \leq m$. Therefore, we have proved that D is a dense right ideal of R. ■

We next study some applications of the results obtained earlier. In Proposition 2.5.5, we have seen that every dense right ideal of a ring R is essential. The following proposition tells us that the converse is also true if R is nonsingular.

Proposition 2.5.7

Let R be a ring. Then R is nonsingular if and only if every essential right ideal of R is dense.

Proof: Assume that R is nonsingular, that is, $Z(R) = \{0\}$. Note that if J is an essential right ideal of R, then $J^l \subseteq Z(R) = \{0\}$. That is, $J^l = \{0\}$ for any essential right ideal J of R. Now let L be an essential right ideal of R. From Proposition 2.3.1(v), we know that $x^{-1}L \; ess \; R$ for any $x \in R$. Therefore $(x^{-1}L)^l = \{0\}$ for all $x \in R$ and hence, L is dense.

Conversely, if $x \in Z(R)$, then $x^r \; ess \; R$ and so x^r is dense in R. In particular, $x^{rl} = \{0\}$. Since $x \in x^{rl}$, we have that $x = 0$. Hence $Z(R) = \{0\}$, as required. ■

We need the following lemma for the proof of the next proposition.

Lemma 2.5.8

Let R be a ring and let $a \in R$ with $a^r = (a^2)^r$. Then

(i) $a^r \cap aR = \{0\}$ and

(ii) $(a^n)^r = a^r$ for all $n \geq 1$.

Proof:

(i) Let $x \in a^r \cap aR$. Then $x = ab$ for some $b \in R$ and $0 = ax = a^2b$. Thus $b \in (a^2)^r = a^r$, which implies that $x = ab = 0$. Therefore, $a^r \cap aR = \{0\}$.

(ii) We prove by induction on n. The result is clearly true if $n = 1$. Suppose that $(a^{n-1})^r = a^r$, where $n > 1$. If $y \in (a^n)^r$, then $ay \in (a^{n-1})^r = a^r$ which implies that

$a^2y = 0$. Therefore, $y \in (a^2)^r = a^r$ and hence, $(a^n)^r \subseteq a^r$. The reverse inclusion $a^r \subseteq (a^n)^r$ clearly holds for all $n \geq 1$. Therefore, $(a^n)^r = a^r$ for all $n \geq 1$. ■

Proposition 2.5.9 (Passman, [Pa3])

Let R be a right finite dimensional ring. If there is a nonzero element $a \in R$ such that $a^r = (a^2)^r$, then $\left(aR \oplus a^r\right)$ *ess* R.

Proof: We first note by Lemma 2.5.8 that a cannot be a nilpotent element of R. Now let I be a nonzero right ideal of R and consider the sum $\sum_{i=1}^{\infty} a^i I$. Since R is right finite dimensional, $\sum_{i=1}^{\infty} a^i I$ cannot be a direct sum of nonzero right ideals of R. There must therefore exist a relation $a^n b_0 + a^{n+1} b_1 + \cdots + a^{n+m} b_m = 0$ for some $n \geq 1$, where $b_i \in I$ and $b_0 \neq 0$. We then have $a^n (b_0 + ab_1 + \cdots + a^m b_m) = 0$, which implies that $b_0 + ab_1 + \cdots + a^m b_m \in (a^n)^r$. Note that

$$b_0 = -a(b_1 + ab_2 + \cdots + a^{m-1} b_m) + (b_0 + ab_1 + a^2 b_2 + \cdots + a^m b_m).$$

Thus $b_0 \in aR \oplus (a^n)^r = aR \oplus a^r$ (by Lemma 2.5.8). Since $b_0 \neq 0$ and $b_0 \in I$, so we have $I \cap \left(aR \oplus a^r\right) \neq \{0\}$. Then since I is arbitrary, we may conclude that $\left(aR \oplus a^r\right)$ *ess* R. ■

Proposition 2.5.10

If R is a right Noetherian ring, then R is right finite dimensional.

Proof: Suppose that there exists an infinite direct sum of nonzero right ideals in R, say $\oplus_{i=1}^{\infty} I_i$. Then $I_1 \subsetneqq I_1 \oplus I_2 \subsetneqq \cdots \subsetneqq I_1 \oplus I_2 \oplus \cdots \oplus I_n \subsetneqq \cdots$ forms a strictly increasing infinite chain of nonzero right ideals in R; a contradiction. Hence, R must be right finite dimensional. ■

Since a right Artinian ring is right Noetherian, the following is an easy consequence of Proposition 2.5.10.

Corollary 2.5.11

If R is a right Artinian ring, then R is right finite dimensional.

Corollary 2.5.12

Let R be a ring without identity. If R is right Noetherian, then there exists an element $a \in R$ such that $\left(aR \oplus a^r\right)$ *ess* R.

Proof: Certainly there must exist an element $a \in R$ such that $a^r = (a^2)^r$ since R satisfies the a.c.c. on right ideals. Hence, the result follows from Propositions 2.5.9 and 2.5.10. ■

Remark: If R is a ring with identity, the conclusion in Corollary 2.5.12 still holds by taking $a = 1$.

One can easily check that a ring R satisfies the a.c.c. on right annihilator ideals if and only if R satisfies the d.c.c. on left annihilator ideals. Suppose that R satisfies the a.c.c. on right annihilator ideals. Let

$$A_1^l \supseteq A_2^l \supseteq \cdots \supseteq A_i^l \supseteq A_{i+1}^l \supseteq \cdots \qquad (*)$$

be a descending chain of left annihilator ideals of R. Then

$$A_1^{lr} \subseteq A_2^{lr} \subseteq \cdots \subseteq A_i^{lr} \subseteq A_{i+1}^{lr} \subseteq \cdots$$

is an ascending chain of right annihilator ideals of R. Since R satisfies the a.c.c. on right annihilator ideals, there exists a positive integer n such that $A_m^{lr} = A_n^{lr}$ for all $m \geq n$. But $A_m \subseteq A_m^{lr} = A_n^{lr}$ implies that $A_n^l A_m = 0$ for all $m \geq n$. It follows that $A_n^l \subseteq A_m^l$ for all $m \geq n$. From (*) we know that $A_m^l \subseteq A_n^l$ for all $m \geq n$. Thus $A_m^l = A_n^l$ for all $m \geq n$. Hence, R satisfies the d.c.c. on left annihilator ideals. The converse can be proved similarly.

Proposition 2.5.13

Let R be a semiprime ring satisfying the a.c.c. on right annihilator ideals. If A, B are right ideals of R so that $B \subseteq A$ and $A^l \subsetneqq B^l$, then there exists an element $a \in A$ such that $aA \neq \{0\}$ and $aA \cap B = \{0\}$.

Proof: We first note that since R satisfies the a.c.c. on right annihilator ideals, so it also satisfies the d.c.c. on left annihilator ideals (hence, the minimum condition on left annihilator ideals). Let I be a left annihilator ideal of R minimal with respect to $A^l \subsetneqq I$ and $I \subseteq B^l$. Then $IA \neq \{0\}$. Since R has no nonzero nilpotent ideals, we have $IAIA = (IA)^2 \neq \{0\}$. Choose $a \in A$ and $x \in I$ so that $IaxA \neq \{0\}$.

We wish to show that $axA \cap B = \{0\}$. Suppose on the contrary that $axA \cap B \neq \{0\}$. Then there exists $y \in A$ such that $0 \neq axy \in axA \cap B$. Since $y \in A$, we have $A^l \subseteq y^l$. Now consider $y^l \cap I$. Observe that $y^l \cap I$ is a left annihilator ideal and $A^l \subseteq y^l \cap I$. Now note that since $axy \in B$ and $I \subseteq B^l$, so $Iaxy = \{0\}$. Therefore $Iax \subseteq y^l \cap I$. Then since $lax \not\subseteq A'$, it follows that $A' \subsetneq y' \cap I$. Thus $y' \cap I = I$ by the minimality of *I*. This implies that $I \subseteq y'$; hence $Iy = \{0\}$ which contradicts the fact that $axy \neq 0$. Therefore, $axA \cap B = \{0\}$. ■

Proposition 2.5.14

Let R be a semiprime ring satisfying the a.c.c. on right annihilator ideals. Let x, y be elements of R such that $xy \neq 0$. If xR *ess* R and yR *ess* R, then xyR *ess* R.

Proof: Let I be a nonzero right ideal of R. Consider $x^{-1}I = \{a \in R \mid xa \in I\}$. Since xR *ess* R, we have $x^{-1}I \neq \{0\}$ and $x(x^{-1}I) = xR \cap I \neq \{0\}$. Then since $xx^r = \{0\}$, so $(x^{-1}I)^l \subsetneq x^{rl}$. It follows from Proposition 2.5.13 that there is a nonzero right ideal J of R such that $J \subseteq x^{-1}I$ and $J \cap x^r = \{0\}$. Consider $y^{-1}J = \{b \in R \mid yb \in J\}$. Clearly, $y(y^{-1}J) \subseteq J$. Since yR *ess* R, we have $y(y^{-1}J) = yR \cap J \neq \{0\}$. Thus $xy(y^{-1}J) \neq \{0\}$. Since $xy(y^{-1}J) \subseteq I$, it follows that $xyR \cap I \neq \{0\}$. Then since I is arbitrary, we may conclude that xyR *ess* R. ■

Proposition 2.5.15

Let R be a semiprime ring satisfying the a.c.c. on right annihilator ideals. If aR *ess* R, then a is not a zero divisor in R.

Proof: We first show that $a^l = \{0\}$. Since $R^l = \{0\}$, so $R^l \subseteq (aR)^l$. If $a^l \neq \{0\}$ (hence, $(aR)^l \neq \{0\}$), then it follows from Proposition 2.5.13 that there exists an element $x \in R$ with $xR \neq \{0\}$ and $xR \cap aR = \{0\}$. This contradicts the fact that aR is essential in R;

whence $a^l = \{0\}$. We now consider a^r. Since R satisfies the a.c.c. on right annihilator ideals, there exists an integer n such that $(a^m)^r = (a^n)^r$ for all $m \geq n$. Then $(a^n)^r = (a^{2n})^r$ and by Lemma 2.5.8, we have $a^n R \cap (a^n)^r = \{0\}$. Since $a^r \subseteq (a^n)^r$, it follows that $a^n R \cap a^r = \{0\}$. Then since $a^n R$ *ess* R (by Proposition 2.5.14), it follows that $a^r = \{0\}$. Hence, a is not a zero divisor in R. ■

Remark: The left analogue of the preceding three propositions have appeared in [He].